Theoretical Computer Science 69 (1989) 319-345
North-Holland

GROWTH PROBLEMS FOR AVOIDABLE WORDS

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Communicated by D. Perrin
Received March 1987
Revised May 1988

Abstract. A word W is said to *avoid* a word U if no block (subword, factor) of W is the image of U under a homomorphism of free semigroups without unit. The theory of words avoiding xx (square-free words) has been much studied. The word U is said to be *avoidable on n letters* if there are arbitrarily long words on an n-letter alphabet that avoid U. If U is avoidable on n letters for some n, let $\mu(U)$ be the minimum possible such n. We show that $\mu(U)$ has a linear bound in terms of the alphabet size of U. We further show that there exists a word that is avoidable on four letters but not on three letters. Moreover, if U is this word, the number of words of length L on a $\mu(U)$-letter alphabet that avoid U has a polynomial bound in terms of L, so that the question of the existence of such an example is resolved in the affirmative. In contrast, for xx the bound is known to be exponential.

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* The work of the third author was partially supported by NSF Grant DMS 85-01969.

1. Introduction

We consider words as finite strings of letters from some alphabet. A problem of increasing interest in the combinatorial theory of words is to determine, for a given pattern of contiguous blocks, the characteristics of words that avoid that pattern. For example, for a given finite alphabet, do there exist arbitrarily long such words? If so, how does the number of such words grow as a function of their length? It is convenient to express the specified pattern by another word, used as a template in a sense to be made precise below.

The oldest related result is due to Thue [31, 32], who showed how to construct arbitrarily long finite words W on a three-letter alphabet that are "square-free", in the sense of having no block that is immediately repeated. For example, the word $W = abcdbcdba$ is not square-free, because its block bcd appears squared, while $W' = abacabcacbabcabacabc$ is square-free. Thus W contains the pattern given by the template word xx, while W' avoids xx. Novikov and Adian used square-free words to solve the Burnside problem [20, 1]; more recently square-free words have been of interest in the study of formal languages [21, 22] and in other algebraic applications [8, 13].

More generally, Bean et al. [3] say that a word W on an alphabet Γ *avoids* a nonempty word U on an alphabet Σ if W has no sequence of contiguous nonempty blocks corresponding in order to the letters of U, such that all occurrences of the same letter in U correspond to blocks of W that are identical as words. In terms of semigroups, if U is on an alphabet Σ and W is on an alphabet Γ, W avoids U if there is no homomorphism h of free semigroups without unit $\Sigma^+ \to \Gamma^+$ such that $h(U)$ is a factor (i.e. a block) of W. If, on the other hand, such an h does exist, let us say that W *encounters* U. (Zimin [34, 35] would say that U *blocks* W.) Thus, for example, the word *abacaba* avoids the word xx, but encounters the word xyx (in several ways).

The word U is simply said to be *avoidable* (or in Zimin's terminology, *non-blocking* [35]) if on some fixed finite alphabet there are arbitrarily long words that avoid U. For example, xx is avoidable but xy is not. The opposite of avoidable is *unavoidable* (or in Zimin's terminology, *blocking* [35]). Bean et al. [3], and independently Zimin [34, 35] show that unavoidable words are those that are reducible to the empty word by a certain recursive algorithmic procedure, rephrased below in Section 2 in an easy graph-theoretic terminology. In other words, avoidability is equivalent to irreducibility. Certain unavoidable words have been used in [23, 24, 30] to characterize those finite semigroups S that are inherently nonfinitely based, in the sense that S is not a member of any locally finite semigroup variety definable by finitely many equations.

A word U is *avoidable on n letters* or *n-avoidable* [3] if for any and hence all alphabets Γ with n letters, there are arbitrarily long words on Γ that avoid U. Results in [31, 32], expressed in these terms, state that xx is 3-avoidable but not 2-avoidable and that xxx is 2-avoidable.

The results of this paper respond to four basic problems in this theory for which relevant previously known results emphasize only the case of words avoiding xx or xxx.

Problem 1. To determine whether there are arbitrarily large positive integers n such that there exists a word that is avoidable but not avoidable on n letters.

Until now, it has not even been known whether there exists a word U that is avoidable but not 3-avoidable. We show that there is such a word, specifically the word *abwbcxcaybazac*, which we denote by U_Δ. (This word is best viewed as *ab w bc x ca y ba z ac*. Because the letters w, x, y, z occur only once, an encounter with U_Δ corresponds to a block pattern $\ldots AB \ldots BC \ldots CA \ldots BA \ldots AC \ldots$. The name U_Δ corresponds to the fact that *ab*, *bc*, *ca*, *ba*, *ac* represent five of the six possible directed sides of a triangle.)

1.1. Theorem. *U_Δ is 4-avoidable but not 3-avoidable.*

The proof is given in Section 7. We further show that if U is any "locked word", of which U_Δ is an example, then U is 4-avoidable (Corollary 5.5). (A word is said to be locked if it cannot be reduced even one step by the recursive algorithmic procedure of [3] and [35].)

Let $\mu(U)$ be the minimum integer n such that U is n-avoidable, or ∞ if U is unavoidable. For example, $\mu(xx)=3$, $\mu(xxx)=2$, and $\mu(xy)=\infty$. Let $\alpha(U)$ be the alphabet size of U (the number of distinct letters appearing in U).

Problem 2. To find an upper bound for $\mu=\mu(U)$ in terms of $\alpha=\alpha(U)$ for avoidable words U.

The papers of Bean et al. [3] and Zimin [35] do not discuss bounds, but implicit in the methods of these papers are the bounds $\mu \leq O(2^{2^{\alpha \log \alpha}})$ and $\mu \leq 96 \cdot 8^\alpha + 3\alpha + 7$, respectively. We present three greatly improved explicit bounds, achieved by two contrasting methods. (Of course, in the unlikely event that Problem 1 has a negative outcome, even these new bounds will have been superseded.) The first of these bounds is of the order of $\alpha \log \alpha$ and is better than the second for $\alpha < 6$.

1.2. Theorem. $\mu < 4(\alpha+2)\lceil \log(\alpha+2) \rceil$, *for all avoidable words U.*

Here $\lceil x \rceil$ denotes the least integer greater than or equal to x (ceiling function). Theorem 1.2 is proved in Section 4, using a construction based on congruences. The second bound is linear:

1.3. Theorem. $\mu < 9 \cdot \alpha + 20$, *for all avoidable words U.*

Theorem 1.3 is proved in Section 6 by a modification of Zimin's construction, which is based on marker letters. Although Theorem 1.3 gives a better bound than Theorem 1.2 for $\alpha \geq 6$, Theorem 1.2 is nevertheless important, in that it leads to a third bound of the same kind in which α is replaced by a quantity that is usually much smaller than the alphabet size. This quantity is the "reductive freedom" of U, denoted $\mathrm{RF}(U)$ and defined in Section 5 in terms of the algorithmic reduction process.

1.4. Theorem. $\mu < 12(R+2)\lceil\log(R+2)\rceil$ *for all avoidable words* U, *where* $R = \mathrm{RF}(U)$.

Theorem 1.4 is proved in Section 5. Because long, complicated pattern words U are apt to have low reductive freedom, Theorem 1.4 accords with the intuition that complicated words ought to be easier to avoid than simpler ones. The method also yields Corollary 5.5, which states that locked words—those words with reductive freedom zero—are 4-avoidable. The contrasting method of Theorem 1.3 does not appear to give such results.

For an n-avoidable word U, it is natural to examine the growth rate of the number of words of length L on an n-letter alphabet that avoid U, as a function of L. The most interesting value of n is $n = \mu(U)$. For example, Brandenburg [6] and Brinkhuis [7] have shown that for $U = xx$, where $\mu(U) = 3$, the growth rate on three letters has an exponential lower bound.

Problem 3. To determine whether there exists an avoidable word U with a less than exponential upper bound on the number of words of length L on a $\mu(U)$-letter alphabet that avoid U, as a function of L.

We show in Section 8 that U_Δ provides an affirmative answer.

1.5. Theorem. *The number of words of length L on a four-letter alphabet that avoid* U_Δ, *has a polynomial upper bound, as a function of L, and a quadratic lower bound.*

Problem 4. For a given word U, to determine topological properties of the space of ω-words or $\mathbb{Z}$-words avoiding U.

Here an ω-word is an infinite sequence of letters indexed by $\omega = \{0, 1, \ldots\}$, and a $\mathbb{Z}$-word is a sequence of letters infinite in both directions, indexed by all integers. Topological properties of the kind mentioned are also related to the general theme of growth, in that they depend on the structure of longer and longer finite words avoiding U. Shelton and Soni [26, 28, 29] proved that the space of ω-words on three letters avoiding xx is perfect. For the case of $U = U_\Delta$, the space of $\mathbb{Z}$-words proves to be most natural, and we obtain the following result, to be proved in Section 9.

1.6. Theorem. *On four letters, the space of* $\mathbb{Z}$*-words avoiding* U_Δ *is perfect, and in fact is a Cantor space.*

Integers will often be used as "letters". By a *block* of a word W we mean a subword of contiguous letters, beginning at a certain position in W. A block is formally describable as a pair $\langle A, \nu\rangle$, where A is the word formed by the letters and ν is the index in W of the first letter of the word. It is usually convenient to be less formal and to refer to the "block A"; if the position of A is relevant, that will be clear from the context.

We shall need an explicit notation for encounters. If W encounters $U = a_0a_1 \ldots a_s$ with corresponding adjacent blocks $A_0A_1 \ldots A_s$ of W, then the encounter can be described by the 4-tuple $\langle U, W, (A_j)_{0\le j\le s}, \nu\rangle$, where ν is the index in W of the first letter of A_0.

By a compactness argument, a word U is avoidable on n letters if and only if there is an ω-word on n letters that avoids U. One ingredient of our method is the construction of particular ω-words $\Omega_{m,z}$ for arbitrary positive integer parameters m and z (Section 3). The ω-word of Dean on four letters [10] is similar to our word $\Omega_{2,1}$; Dean's ω-word is constructed to be square-free on symbols x, y, x^{-1}, y^{-1}, with no opportunities for group-theoretic cancellation.

Recent and interesting contributions to the study of avoidable words are found in [25] and [17], where further applications to semi-groups are considered. The book by Lothaire [15] and the conference proceedings edited by Cummings [9] are general references for ideas related to this paper.

Kobayashi [14] discusses an even more general concept of avoidance in which a set of specific words is to be avoided rather than all substitutions of blocks for letters of a given "pattern" word, as in this paper. The set of all $\mathbb{Z}$-indexed words avoiding a given finite set of words in Kobayashi's sense is a subshift of finite type, in the terminology of symbolic dynamics [11], while avoidance of patterns in our sense requires an infinite set of specific words [14, p. 180].

2. Reducibility of words

Let us review the Bean–Ehrenfeucht–McNulty–Zimin characterization of avoidable words [3, 34, 35] in graph-theoretic language.

Let U be a word on Σ. With U we associate a bipartite graph $\mathrm{AG}(U)$, the *adjacency graph* of U: The vertex set consists of two copies of Σ, namely Σ^{R} and Σ^{L} (R, L for right and left); for each $x \in \Sigma$ there is a vertex $x^{\mathrm{R}} \in \Sigma^{\mathrm{R}}$ and a vertex $x^{\mathrm{L}} \in \Sigma^{\mathrm{L}}$. There is an edge x^{R}—y^{L} for $x, y \in \Sigma$ whenever xy occurs as a block of U; this edge signifies that the "right side" of x is adjacent to the "left side" of y. Figure 1 shows two examples of adjacency graphs; Fig. 2 in Section 7 shows a third.

A nonempty subset F of Σ is said to be *free for U* if no connected component of $\mathrm{AG}(U)$ contains both an element of F^{R} and an element of F^{L} (the obvious two copies of F). Although we shall not make use of it, one intuitive interpretation of the relevance of components is that if, say, there is a path a^{R}—b^{L}—c^{R}—d^{L} in $\mathrm{AG}(U)$, so that U has blocks ab, cb, and cd, then a word W encountering U has corresponding blocks AB, CB, and CD; if blocks are regarded as puzzle pieces whose ends can in some sense fit together, then from the fact that the right-hand

(a) AG($abcba$) (b) AG($abcdadcb$)

Fig. 1.

end of A fits the left-hand end of B and from the similar fittings of C and B and of C and D, we deduce that the right-hand end of A fits the left-hand end of D. A set F is a free set for U when the 2-blocks of U do not mandate any fittings between images of the members of F in words W. For example, in Fig. 1(a) the free sets are $\{a\}$, $\{b\}$, $\{c\}$, $\{a, c\}$; in Fig. 1(b) the free sets are $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{b, d\}$; and in Fig. 2 in Section 7 the graph is connected and there are no free sets.

We say that U *reduces to* U' *in one step* if U' can be obtained from U by deleting from U all occurrences of the letters of some set free for U. We say that U *reduces to* V if V is obtainable from U by successive one-step reductions. Finally, we say that U is *reducible* if U reduces to the empty word, and *irreducible* otherwise.

Even if a word is reducible, it may be that some one-step reductions do not lead to the empty word. For example, the word $abcba$ of Fig. 1(a) can be reduced to the empty word by deleting in order a, b, c or b, a, c but not by deleting c or a, c first, even though $\{c\}$ and $\{a, c\}$ are free sets. Moreover, Bean et al. give an example of a word that can be reduced only if more than one letter is deleted at a time [3]. In Fig. 1(b), there are one-step reductions, but the word is irreducible: As observed by Bean et al. [3] and Zimin [35], in a reducible word at least one letter appears only once, because the last free set to be deleted must be a singleton.

2.1. Theorem (Bean et al. [3, Theorem 3.22], Zimin [35, Theorem 1]). *A word is avoidable if and only if it is irreducible.*

In [3] and [35] there are inductive proofs that every reducible word is unavoidable, and we offer no improvement. Our Theorem 4.1 constitutes a strengthening of the converse.

3. The infinite words $\Omega_{m,z}$

Our main results depend on certain ω-words $\Omega_{m,z}$ determined by integer parameters $m \geqslant 2$ and $z \geqslant 1$. These infinite words have such regular properties that an analysis of their encounters with finite words is feasible. The word $\Omega_{m,z}$ is based on the alphabet $\{0, \ldots, n-1\}$ for $n = m\lceil m^{1/z}\rceil$.

The ω-word $\Omega_{m,z}$ is constructed via an endomorphism, with images of generators having length m. For example, $\Omega_{2,1}$ is constructed from the endomorphism $h_{2,1}$ given by $0\to 01$, $1\to 21$, $2\to 03$, $3\to 23$, on the free semigroup Γ^+, where $\Gamma=\{0,1,2,3\}$. The resulting infinite word is the common extension of 0, $h(0)$, $h^2(0),\ldots$, namely, $\Omega_{2,1}=0121032101230\ldots$, a variant of the infinite word of Dean [10].

3.1. Construction of $\Omega_{m,z}$. Let any $m>1$ and $z>0$ be given.

Let $n=m\lceil m^{1/z}\rceil$. We first construct an endomorphism $h_{m,z}$ of the free semigroup Γ^+ without unit, where $\Gamma=\{0,1,\ldots,n-1\}$, as follows. For convenience let $b=\lceil m^{1/z}\rceil$, so that $m\leq b^z$, $n=bm\leq b^{z+1}$. Then each $i=0,\ldots,n-1$ has a $(z+1)$-digit representation $d_z^i d_{z-1}^i\ldots d_0^i$ to the base b. Writing these digits in reverse order, extend the list cyclically to obtain a sequence of length m: $d_0^i d_1^i\ldots d_{m-1}^i = d_0^i d_1^i\ldots d_z^i d_0^i d_1^i\ldots$, ending with the subscript that is the residue of $m-1$ modulo $z+1$. Let h be defined for generators i, $0\leq i<n$, by $h(i)=e_0^i e_1^i\ldots e_{m-1}^i$, where for $0\leq j<m$, $e_j^i=md_j^i+j$.

Now define words $W_0, W_1,\ldots$ by $W_k=h^k(0)$. Because $h(0)$ starts with 0, each of the words W_0, W_1, $W_2,\ldots$ extends the preceding and so all are initial segments of a uniquely determined ω-word $\Omega_{m,z}$.

For example, in the construction of $\Omega_{5,3}$, we have $m=5$, $z=3$, $b=2$, $n=10$; $h_{5,3}$ is as indicated in Table 1.

The properties of $\Omega_{m,z}$ to be used in succeeding sections are derived from several properties of $h_{m,z}$.

Table 1. $h_{5,3}$.

i	$d_z^i\ldots d_0^i$	$d_0^i d_1^i\ldots$ (cyclically)	$md_0^i md_1^i\ldots$	$h_{m,z}(i)=e_0^i\ldots e_{m-1}^i$
0	0000	00000	00000	01234
1	0001	10001	50005	51239
2	0010	01000	05000	06234
3	0011	11001	55005	56239
4	0100	00100	00500	01734
5	0101	10101	50505	51739
6	0110	01100	05500	06734
7	0111	11101	55505	56739
8	1000	00010	00050	01284
9	1001	10011	50055	51289

3.2. Lemma. *For any $m>1$ and $z>0$, the endomorphism $h=h_{m,z}$ of the free semigroup Γ^+, where $\Gamma=\{0,1,\ldots,n-1\}$, $n=m\lceil m^{1/z}\rceil$, has these properties for each $i<n$:*

(i) *the length of $h(i)$ is m;*

(ii) *the residues modulo m of the letters of $h(i)$ are $0, 1,\ldots, m-1$, in that order;*

(iii) *i can be recovered from a knowledge of any $z+1$ consecutive entries of $h(i)$. Furthermore,*

(iv) *$h(0)$ begins with 0.*

Proof. The properties (i), (ii), and (iv) are immediate; (iii) follows from the observation that from any $z+1$ consecutive entries of $h(i)$, $d_0^i d_1^i \ldots d_{z-1}^i$ can be recovered. □

Observe that by (i) and (ii) of Lemma 3.2, each W_k reduces modulo m to a periodic sequence, and therefore so does $\Omega_{m,z}$. In other words, the residue modulo m of a letter of $\Omega_{m,z}$ is the same as the residue modulo m of its index (position) in $\Omega_{m,z}$, where the initial letter has index 0. This fact helps to regulate the kinds of encounters that $\Omega_{m,z}$ can have. For an encounter $E=\langle U, \Omega_{m,z}, (A_j)_{0\le j\le s}, i\rangle$, by the *left boundary residue* of a block A_j let us mean the residue modulo m of the first letter of A_j. By the *right boundary residue* of A_j let us mean the residue modulo m of 1 plus the last letter of A_j. For brevity, by a *crack* of the encounter E let us mean simply any boundary residue, left or right. Thus two adjacent blocks share a common crack. Even though cracks are defined as residues, in working with them we always treat them as members of the ordered set $\{0, 1, \ldots, m-1\}$, non-cyclically (i.e. not regarding 0 as following $m-1$).

The next lemma gives a collection of useful facts about cracks. For $c \in \{0, \ldots, m-1\}$ let us say that c is *left-predictable* if whenever A_j and $A_{j'}$ are blocks that are identical as words and have left boundary residue c, then A_j and $A_{j'}$ are preceded by the same letter. Similarly, let us say that c is *right-predictable* if whenever A_j and $A_{j'}$ are blocks that are identical as words and have right boundary residue c, then A_j and $A_{j'}$ are followed by the same letter. (Because of the increment of 1 in the definition of the right boundary residue, that letter will have residue c.)

3.3. Lemma. *Consider an encounter $E=\langle U, \Omega_{m,z}, (A_j)_{0\le j\le s}, \nu\rangle$.*

(a) *Each connected component K of $\mathrm{AG}(U)$ is associated with a single crack c, in the sense that for each $a_j \in K\cap \Sigma^{\mathrm{R}}$, A_j has left boundary residue c, and for each $a_{j'} \in K \cap \Sigma^{\mathrm{L}}$, $A_{j'}$ has right boundary residue c.*

(b) *If $A_{j(1)}, \ldots, A_{j(q)}$ are such that the set of their left boundary residues is disjoint from the set of their right boundary residues, then $\{a_{j(1)}, \ldots, a_{j(q)}\}$ is a free set.*

(c) *Suppose c is a crack with $0<c<m-z$. If $c+1, c+2, \ldots, c+z$ are noncracks, then c is left-predictable. More generally, if all blocks A_j with left boundary residue c are of length at least $z+1$, then c is left-predictable.*

(d) *Suppose c is a crack with $z<c<m$. If $c-z, c-z+1, \ldots, c-1$ are noncracks, then c is right-predictable. More generally, if all blocks A_j with right boundary residue c are of length at least $z+1$, then c is right-predictable.*

Proof. (a) and (b) are immediate from Lemma 3.2(i), and (c) and (d) from Lemma 3.2(iii). □

In (d), note the reason that the assertion cannot be extended to include the case $c=z$: If A_j is a block with right boundary residue $c=z$, then the rightmost letter of A_j has residue $z-1$, and only z letters of A_j lie in a single $h(i)$.

4. The proof of Theorem 1.2

Recall the statement of Theorem 1.2: $\mu<4(\alpha+2)\lceil\log(\alpha+2)\rceil$, for all avoidable words U, where α is the alphabet size of U and μ is the least n such that U is n-avoidable. Much of this section will be devoted to proving a fact from which Theorem 1.2 follows readily:

4.1. Theorem. *For integers r, m, z with r, $z>0$ and $m>(r+1)z$, the ω-word $\Omega_{m,z}$ on $n=m\lceil m^{1/z}\rceil$ letters avoids all irreducible words on r or fewer letters.*

For each r, an obvious goal would be to choose the parameters z and m to yield the minimum possible n. However, such choices do not follow an easily described pattern in terms of r. Nevertheless, the choices $z=\lceil\log(r+2)\rceil$, $m=z(r+2)$ give a satisfactory approximation to the minimum n, the bound asserted in Theorem 1.2:

Proof of Theorem 1.2 from Theorems 2.1 and 4.1. It is a fact that for all positive integers z, $z^{1/z}<4/\mathrm{e}$. In Theorem 4.1, for given r choose $z=\lceil\log(r+2)\rceil$ and $m=z(r+2)$. Then $\log(r+2)/z\leqslant 1$ and hence $(r+2)^{1/z}=\mathrm{e}^{\log(r+2)/z}\leqslant\mathrm{e}$. Therefore $n=m\lceil m^{1/z}\rceil=m\lceil z^{1/z}(r+2)^{1/z}\rceil\leqslant m\lceil(4/\mathrm{e})\mathrm{e}\rceil=4m=4(r+2)z$. Thus each avoidable word U with $\alpha(U)\leqslant r$, being irreducible by the "only if" part of Theorem 2.1, is avoidable on at most $4(r+2)z=4(r+2)\lceil\log(r+2)\rceil$ letters, as claimed in Theorem 1.2. □

4.2. Remark. If $\log(r+2)>36$, so that $z\geqslant 37$, it is a fact that $z^{1/z}<3/\mathrm{e}$; then the bound becomes $n<3(r+2)z$.

The proof of Theorem 4.1 occupies the balance of this section. We now assume that the parameters r, z, m have the relationships stated in the theorem, and we fix an r-letter alphabet. Let Enc be the set of encounters of $\Omega_{m,z}$ with finite words on this alphabet.

Three relevant measures of an encounter $E=\langle U,\Omega_{m,z},(A_j)_{0\leqslant j\leqslant s},\nu\rangle\in\mathrm{Enc}$ are these: (1) $\alpha(U)$, the number of distinct letters in U; (2) $\kappa(E)$, the minimum k such that the blocks A_j are all contained in $W_k\subseteq\Omega_{m,z}$; (3) $\mathrm{CNC}(E)$, the maximum length of a stretch of consecutive noncracks among $1,\ldots,m-1$. (Here "consecutive" is *not* interpreted cyclically.) We compare encounters by these three measures, but with the third reversed. Specifically, we compare such E by lexicographically ordering the associated triples of numbers $(\alpha(U),\kappa(E),m-\mathrm{CNC}(E))$.

To prove Theorem 4.1, we show that any such encounter E of $\Omega_{m,z}$ with a nonempty word U on the fixed r-letter alphabet leads algorithmically to a lexicographically smaller encounter $E' = \langle U', \Omega_{m,z}, (A'_j)_{0 \le j \le s'}, \nu' \rangle \in \mathrm{Enc}$ such that U' is either U itself or a one-step reduction of U. (Repetition of this process then results in a reduction of $\Omega_{m,z}$ to the empty word.) There are three cases, the last proved with the lemmas needed interspersed.

Case 1: The first letter a_0 of U is not the same as any other letter, or the last letter a_s is not the same as any other letter. (In either instance let us say that U has *a dangling letter.*)

A letter appearing only at the beginning of U or only at the end constitutes a free set as a singleton. If such a letter is deleted from U to obtain a word U' and the corresponding block A_0 or A_s is deleted from the encounter with $\Omega_{r,z}$, a lexicographically smaller encounter E' is obtained. Thus Case 1 is verified.

Case 2: E has no nonzero cracks.

In this case, modulo m each block A_j has first letter 0 and last letter $m-1$. Then for $h = h_{m,z}$ we may apply h^{-1} to the A_j to obtain a lexicographically smaller encounter $E' = \langle U, \Omega_{m,z}, h^{-1}(A_j)_{0 \le j \le s}, \nu' \rangle$, with $\kappa(E') = \kappa(E) - 1$.

Case 3: E has a nonzero crack but no dangling letter.

In this case, we use a combinatorial fact stemming from the relation of z and m to r, as follows. (For future reference, the statement assumes only that $m \ge (r+1)z$ rather than the hypothesis $m > (r+1)z$ of Theorem 4.1, under which possibility (iii) cannot occur.)

4.3. Lemma. *Let E be an encounter of $\Omega_{m,z}$ with U on r letters, where $m \ge (r+1)z$. Suppose there is a nonzero crack and no dangling letter. Then one or more of these possibilities holds*:

(i) *there exists a nonzero left-predictable crack followed by* CNC(E) *consecutive noncracks*;

(ii) *there exists a nonzero right-predictable crack preceded by* CNC(E) *consecutive nonzero noncracks*;

(iii) *$m = (r+1)z$ and the cracks are precisely $z, 2z, \ldots, rz$.*

Proof. Let $c+1, c+2, \ldots, d-1$ in $\{1, \ldots, m-1\}$ be a stretch of nonzero consecutive noncracks of maximum length (namely, of length CNC(E)), as usual with "consecutive" *not* interpreted cyclically. Thus c is a crack or 0, and d is a crack or m. In fact, because there is a nonzero crack, either $c > 0$ or $d < m$ or both. Suppose that $c > 0$ and $d - c \ge z+1$. Then c is left-predictable by Lemma 3.3(c). Suppose that $d < m$ and $d - c \ge z+1$. Then d is right-predictable by Lemma 3.3(d). The remaining possibility is that $d - c \le z$, or in other words that no stretch among $1, \ldots, m-1$ has more than $z-1$ noncracks. Because U is a word on at most r letters, there are at most r blocks A_j that are distinct as words, hence at most r cracks (because there are no dangling letters), hence at most $r+1$ stretches without cracks; therefore we have $m - 1 \le (r+1)(z-1) + r = (r+1)z - 1$ and since $m \ge (r+1)z$, the only possibility is equality, with configuration (iii). □

Proof of Theorem 4.1 (*continued*). Suppose $c>0$ is a left-predictable crack followed by CNC(E) noncracks, as in (i) of Lemma 4.3. Let us construct a proposed encounter $E'=\langle U', \Omega_{m,z}, (A'_j)_{0\le j\le s'}\nu'\rangle$ by "shifting E left at c", as follows.

(1) Delete any blocks A_j that are singletons with sole letter having residue $c-1$ modulo m, and delete the corresponding letters in U to obtain U'.

(2) From each remaining block A_j whose last letter is $c-1$ modulo m delete that last letter.

(3) To each remaining block A_j whose first letter is c modulo m, prepend the letter immediately to its left.

(4) Renumber the letters of U' and correspondingly, the remaining blocks, to obtain E'.

Observe that E' really is an encounter, because c is left-predictable. Moreover, by Lemma 3.3(b) with $q=1$, the letters of U deleted in step (1), when E is shifted right or left, form a free set, and U' is therefore a one-step reduct of U.

If, on the other hand, d is a right-predictable crack preceded by CNC(E) consecutive noncracks, as in (ii) of Lemma 4.3, then the E' can be similarly defined by shifting E right: singleton blocks with entry having residue d are deleted, and boundaries where remaining blocks have last letter $d-1$ or first letter d modulo m are shifted right one.

The proof of Case 3 is now completed by observing that $m-\text{CNC}(E')<m-\text{CNC}(E)$, so that E' is indeed lexicographically smaller than E. □

There is one case in which the size of m in Theorem 4.1 can be lowered by 1.

4.4. Theorem. *For any integers r, m, z with $r\ge 0$ and m, $z>0$, if*

(a) $m\ge(r+1)z$,

(b) $z+1$ *divides* m, *and*

(c) $\Omega_{m,z}$ *has the property that its* qth *letter is the residue of q modulo $n=m\cdot\lceil m^{1/z}\rceil$ for each q divisible by $z+1$,*

then $\Omega_{m,z}$ avoids every irreducible word on at most r letters.

Proof. The proof is the same as for Theorem 4.1, except that the possibility (iii) of Lemma 4.3 can occur. In this case, the possibility (i) also occurs, for the following reason. Under (iii), because 0 is not a crack, any block A_j with left boundary residue $m-z=rz$ includes a letter with residue 0 modulo m. The index of this letter is divisible by $z+1$, as is the index of the letter preceding A_j. Therefore by (c) the former letter determines the latter. □

4.5. Example. Theorem 4.4 applies to $\Omega_{m,1}$ for each even m. The reasoning is as follows. In Construction 3.1, $b=\lceil m^{1/1}\rceil=m$ and $n=m^2$. If we write each i with $0<i<n$ to the base $b=m$ as $i=[d^i_1d^i_0]$, using this notation as if it were a single letter, then $h(i)=h([d^i_1d^i_0])=[d^i_10][d^i_01][d^i_12][d^i_03]\dots[d^i_0(m-1)]$. Then by applying the same rule to the "letters" of $h(i)$ using their base $b=m$ representations we

obtain

$$h(h(i)) = [00][d_1^i 1][02][d_1^i 3]\dots[0(m-2)][d_1^i(m-1)][10][d_0^i 1][12][d_0^i 3]\dots$$
$$\dots[1(m-2)][d_0^i(m-1)]\dots[(m-1)0][d_0^i(m-1)]\dots$$
$$\dots[(m-1)(m-2)][d_0^i(m-1)],$$

where the even-subscripted letters are the base $b=m$ representations of $0,2,4,\dots,n-2$ and so are even. Because $\Omega_{m,1}$ is obtained as a union of words $h^k(0)$, even-subscripted letters of $\Omega_{m,1}$ must be even, as required.

5. The reductive freedom of a word

The proof of Theorem 4.1 depended on the value of r only in the proof of Lemma 4.3. In that proof, the disparity between r and m was used, in conjunction with the pigeon-hole principle, to start the revision process. We shall show that in many cases the proof remains valid when the parameter r is chosen to be considerably smaller than the alphabet size of U.

By a *reduct* of a word U let us mean any word U' obtained from U by recursively deleting free sets. Consider the cardinalities of the free sets occurring in U and its reducts. Let us call the maximum such cardinality the *reductive freedom* of U, denoted $\mathrm{RF}(U)$. Evidently $\mathrm{RF}(U) \leqslant \alpha(U)$, the alphabet size of U. The example of Fig. 1(b) has reductive freedom 2 and is irreducible.

5.1. Theorem. *Let r, m, z be integers with $r \geqslant 0$, $z > 0$, and $m > 2(r+1)z$. Then the word $\Omega_{m,z}$ on $n = m\lceil m^{1/z}\rceil$ letters avoids every irreducible word U with* $\mathrm{RF}(U) \leqslant r$.

The proof is given below. Recall the statement of Theorem 1.4: $\mu < 12(R+2)\lceil\log(R+2)\rceil$ for all avoidable words U, where $R = \mathrm{RF}(U)$ and μ is the least n such that U is n-avoidable. As with the proof of Theorem 1.2 from Theorem 4.1, Theorem 1.4 follows readily from Theorem 5.1, but starting from the relation $m \geqslant 2z(r+2)$.

5.2. Remark. Analogously to Remark 4.2, in Theorem 1.4 if $\log(R+2) > 46$ then $\mu \leqslant 6(R+2)\lceil\log(R+2)\rceil$.

In preparation for a proof of Theorem 5.1, let us note the following fact as a substitute for Lemma 4.3. (Again, (iii) cannot occur under the hypotheses of Theorem 5.1.)

5.3. Lemma. *Let E be an encounter of $\Omega_{m,z}$ with U, where* $\mathrm{RF}(U) \leqslant r$ *and* $m \geqslant 2(r+1)z$. *Suppose E has a nonzero crack and no dangling letters. Then one or more of these possibilities holds*:

(i) *E has a crack among $1,\dots,z$ and there exists a nonzero left-predictable crack*;

(ii) *E has no crack among $1, \ldots, z$ and there exists a nonzero right-predictable crack;*
(iii) *$m = 2(r+1)z$ and the cracks are precisely $z, 2z, \ldots, (2r+1)z$.*

Proof. If E has no crack among $1, \ldots, z$ then the first nonzero crack is right-predictable by Lemma 3.3(d), and (ii) holds. From now on, then, let us suppose that there is a crack $c(1)$ among $1, \ldots, z$. Because there are no dangling letters, every crack is the left boundary residue of some block. If any crack c with $0 < c < m-1$ is such that all blocks A_j with left boundary residue c are of length at least $z+1$, then c is left-predictable by Lemma 3.3(c), and so (i) holds. Let us suppose, then, that every nonzero crack $c < m - z$ is the left boundary residue of some block $A_{j(c)}$ of length at most z. We can then construct a sequence of cracks $c(1), c(2), \ldots$ by recursively letting $c(i)$ be the right boundary residue of $A_{j(c(i-1))}$ for each $i > 1$. Inductively, $c(i) \leqslant c(0) + (i-1)z \leqslant iz$ for each i. Because $m \geqslant 2(r+1)z$, we may continue at least as far as $c(2r+1)$, the right boundary residue of $A_{j(2r)}$. In fact, if $c(1) < z$ or if $m > 2(r+1)z$ or if any of the blocks $A_{j(c(i))}$ have length $<z$, we may continue to $c(2r+2)$, the right boundary residue of $A_{j(2r+1)}$. But then among the $r+1$ blocks $A_{j(1)}, A_{j(3)}, A_{j(5)}, \ldots, A_{j(2r+1)}$ no left boundary residue coincides with a right boundary residue, and by Lemma 3.3(b), the $r+1$ letters $a_{j(1)}$, $a_{j(3)}$, $a_{j(5)}, \ldots, a_{j(2r+1)}$ form a free set, contrary to hypothesis. The only possibility remaining is that $c(1) = z$, $m = 2(r+1)z$, and all of the blocks $A_{j(c(i))}$ $(i = 1, \ldots, 2r)$ have length z. Then at least $z, 2z, \ldots, (2r+1)z$ are cracks. Let us show that there are no more cracks and thereby verify (iii). Suppose that d is an additional crack, the left boundary residue of a block $A_{j'}$. In order for the $r+1$ blocks $A_{j'}, A_{c(1)}$, $A_{c(3)}, \ldots, A_{c(2r-1)}$ not to produce a free set of $r+1$ letters by Lemma 3.3(b), some left boundary residue must coincide with some right boundary residue, so that the right boundary residue of $A_{j'}$ must be among $2z, 4z, \ldots, 2rz$. The same argument for the blocks $A_{j'}, A_{c(2)}, A_{c(4)}, \ldots, A_{c(2r)}$ shows that the right boundary residue of $A_{j'}$ must be among $3z, 5z, \ldots, (2r+1)z$, a contradiction. $\square$

With Lemma 5.3 substituted for Lemma 4.3, the proof of Theorem 5.1 is now the same as the proof of Theorem 4.1, except that the incorporation of CNC(E) in the lexicographic measure of the size of an encounter is no longer relevant for the proof. Instead, we can measure an encounter simply by (α, κ) and in Case 3 use this procedure: Repeatedly, as long as there is any crack among $1, \ldots, z$, find a nonzero left-predictable crack and shift it left. This action reduces the sum of all cracks and so we must end with no crack among $1, \ldots, z$. Next, as long as there is any nonzero crack, repeatedly find a nonzero right-predictable crack and shift it right. This action reduces the sum of $m - c$ over all nonzero cracks c and so we must end with no nonzero crack.

As before, there is one useful case in which the minimum size of m in Theorem 5.1 can be lowered by 1.

5.4. Theorem. *For any integers r, m, z with $r \geqslant 0$ and $m, z > 0$, if*

(a) $m \geqslant 2(r+1)z$,

(b) $z+1$ *divides* m, *and*

(c) $\Omega_{m,z}$ *has the property that its* qth *letter is the residue of* q *modulo* $n = m \cdot \lceil m^{1/z} \rceil$ *for each* q *divisible by* $z+1$,

then $\Omega_{m,z}$ *avoids every irreducible word* U *with* $\mathrm{RF}(U) \leqslant r$.

The method of proof is identical to that of Theorem 4.4, with case (iii) of Lemma 5.3 in place of (iii) of Lemma 4.3.

We shall refer to words U with $\mathrm{RF}(U)=0$ as *locked* words. Locked words are those that have no nonempty free sets. Such a word cannot be reducible, since there is no way even to begin a reduction process. By taking $r=0$, $z=1$ in Theorem 5.4, we obtain this fact:

5.5. Corollary. *The word $\Omega_{2,1}$ on four letters avoids every locked word.*

Proof. By Example 4.5, Theorem 5.4 applies to $\Omega_{2,1}$. □

5.6. Example. Bean et al. [3] call a word U *scrambled* provided every letter that occurs in U occurs at least twice in U and, moreover, if x and y are distinct letters occurring in U, then both xy and yx are blocks of U. They proved that every scrambled word is avoidable on a twenty-letter alphabet. Evidently, every scrambled word on three or more letters is locked and so is 4-avoidable. (Those on at most two letters are 3-avoidable.)

5.7. Remark. $\Omega_{2,1} = 0121032101230321\ldots$ does encounter some irreducible words. For example, the word $abcdadbc$ of Fig. 1(b) is irreducible, but with the correspondence $A=0$, $B=1$, $C=2$, $D=3$ this word encounters $\Omega_{2,1}$ starting at the third occurrence of 0. Further analysis shows that any infinite word on a finite alphabet must encounter some irreducible word, in fact, one of its own blocks: According to [3, Theorem 3.13] and [35, Lemma 12], any block U of length $2^{\alpha(U)}$ or greater is avoidable and hence irreducible.

6. The linear bound

Theorem 1.3 states a linear bound $\mu(U) \leqslant 9 \cdot \alpha(U) + 20$ for avoidable words U, where $\alpha(U)$ is the alphabet size of U and $\mu(U)$ is the least n such that U is n-avoidable. We obtain this bound by modifying in two ways the method of Zimin [35, Lemmas 9, 10, 11, and the proof of Theorem 1]. First, by a simple graph-theoretic argument we increase the efficiency of his basic construction. Second, we observe that in key parts of his proof the use of $l(U)$, the length of U, can be replaced by the use of $\alpha(U)$, the alphabet size of U. Insofar as possible, we shall refer to [35] for details that remain unchanged.

6.1. Lemma (improving [35, Lemma 9]). *For any integer $r \geqslant 2$ there is a set of $4r-4$ words of length r, on an alphabet of only $2r-1$ letters, such that*

(a) *any two distinct words have no common subwords of length* 2;

(b) *each letter occurs in each word at most once.*

Proof. Consider the complete graph with $2r-1$ vertices $a_1, \ldots, a_{2r-1}$. A standard result in graph theory [4, Corollary 1, p. 233] states that there are $r-1$ pairwise edge-disjoint Hamiltonian cycles. Each such cycle can be regarded as a word of length $2r-1$. From each of these $r-1$ words, make four words of length r by taking (i) the first r letters, (ii) the last r letters, (iii) the first r letters in reverse order, and (iv) the last r letters in reverse order. The resulting $4r-4$ words of length r have the desired properties. □

Now assume $r \geqslant 3$ and let $k = 2r-1$, so that $r+k \leqslant 4r-4$. Discard all but $r+k = 3r-1$ of the words constructed in Lemma 6.1. As in [35], introduce r new letters $b_1, \ldots, b_r$ and alternate them with the letters of the words not discarded, to obtain $3r-1$ words of length $2r$ of the form $a_{i(1)}b_1a_{i(2)}b_2 \ldots a_{i(r)}b_r$, on a combined alphabet Σ of $3r-1$ letters. Zimin [35] calls such words *w-words* and concatenations of such words, *whole words*. The letters b_j serve as markers to enable recovery of positional information within words, much as residues modulo m served in the construction of Sections 3–5.

The next two lemmas are directly from [35], except that $\alpha(P)$ is used in place of the length $l(P)$ and terminology from this paper is used. The proofs remain unchanged otherwise and are omitted.

6.2. Lemma (improving Zimin [35, Lemma 10]). *Suppose for a word P there is fixed a system of w-words of length at least $2 \cdot \alpha(P)+2$. If P is encountered by a word W in such a way that the image of P in W is a whole word and each individual block of the encounter is a whole word or block of a w-word, then for some reduct Q of P, there is an encounter of W with Q in which each block is a whole word.*

6.3. Lemma (improving Zimin [35, Lemma 11]). *Suppose for a word P there is fixed a system of w-words of length at least $6 \cdot \alpha(P)+14$. If P is encountered by a whole word W then for some reduct Q of P, there is an encounter of W with Q in which each block is a whole word.*

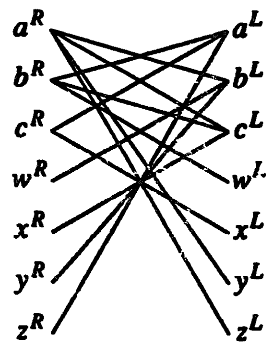
Now, to prove Theorem 1.3, Zimin's argument based on Lemma 6.3 remains valid, whereas in Lemma 6.3 the w-words are chosen to have length $6 \cdot \alpha(P)+14$, i.e. $2r = 6 \cdot \alpha(P)+14$. Thus the w-words considered are on an alphabet of $3r-1 = 9 \cdot \alpha(P)+20$ letters. If τ is any one-to-one correspondence of the $3r-1$ letters with the $3r-1$ w-words used, then τ extends to an endomorphism $\tau : \Sigma^+ \to \Sigma^+$ whose iterates applied to a single letter yield arbitrarily long words avoiding P. By a compactness argument, such words produce an infinite word avoiding P. (Indeed, if the w-words are constructed instead as $b_1 a_{i(1)} b_2 a_{i(2)} \dots$, then b_1, $\tau(b_1)$, $\tau^2(b_1), \dots,$ extend each other to produce an infinite word directly.)

7. The unavoidability of U_Δ on three letters

The word $U_\Delta = abwbcxcaybazac$

As remarked in Section 1, U_Δ is best viewed as $ab\ w\ bc\ x\ ca\ y\ ba\ z\ ac$, and because w, x, y, z each occur only once in U_Δ they serve only as "place holders". That is to say, if W is a word for which a, b, and c can be mapped to words A, B, C in the alphabet of W so that W contains blocks AB, BC, etc., in order and separated by one or more letters, then W encounters U_Δ.

7.1. Observation. U_Δ is a locked word. It suffices to examine the adjacency graph, shown in Fig. 2.



AG(U_Δ), $U_\Delta = abwbcxcaybazac$

Fig. 2.

To study the words that avoid a given word, it is useful to indicate their transitions (2-blocks) in the form of a directed graph, rather than in the form of the bipartite adjacency graph.

7.2. Definition. For a word or ω-word W, the *transition digraph* TDG(W) is the directed graph whose vertices are the letters appearing in W and whose edges are all the 2-blocks appearing in W. TDG(W) may have loops but not multiple loops or edges.

For example, $\Omega_{2,1} = 0121032101230\dots$ has the transition digraph C_4 (Fig. 3), while the word 21202102012102 of length 14 has the transition digraph C_3. Here a double arrow denotes a pair of edges, one in each direction.

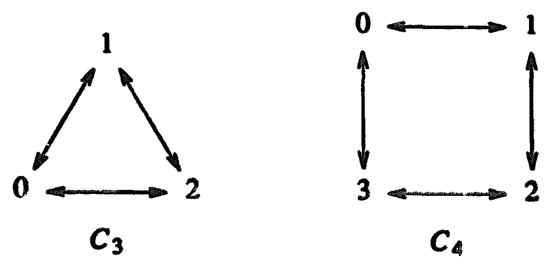


Fig. 3.

7.3. Lemma. *Each square-free word of length* 14 *or more on three letters* $\{0, 1, 2\}$ *has transition digraph* C_3.

Proof. Suppose, to the contrary, that the word omits some 2-block, say 02. If also 20 does not occur, then the word is easily seen to be a block of one of 0121012, 1012101, 1210121, 2101210, 010, 212, for a length at most seven. If instead 20 does occur, then the word can be broken into two or more blocks in which 20 does not occur, so that all blocks except the first start with 0 and all blocks except the last end with 2. A block that *both* starts with 0 and ends with 2 must be 012 or 0121012; therefore to avoid having 012012, a square, there can be at most one block other than the first and last. If there is one, the longest possible first and last blocks are 212 and 010, for a maximum word length of 13. If there are only two blocks, they cannot both be 0121012, so that one has length six or less, again for a maximum length of 13. □

Proof of Theorem 1.1. This theorem had two assertions: (a) that U_Δ is 4-avoidable and (b) that U_Δ is not 3-avoidable. For (a), by Observation 7.1, U_Δ is locked and hence 4-avoidable by Corollary 5.5. For (b), we proceed as follows. Let us use the three-letter alphabet $\{0, 1, 2\}$. Consider a long word W, and starting at the left mark off consecutive blocks of lengths 14, 1, 14, 1, ... By Lemma 7.3, each block of 14 letters must either (i) contain a square or (ii) contain all six two-letter pairs. Here are two cases in which W must encounter U_Δ: (1) If some five of the 14-letter blocks are of type (i) with identical squares (for example, if all five contain 021021), then let $A = B = C =$ the common block that appears squared (021). (2) If some five of the 14-letter blocks are of type (ii) then let $A = 0$, $B = 1$, $C = 2$. In both cases, observe that AB can be found in the first block of the five, BC in the second, and so on for CA, BA, AC. Then W encounters U_Δ. (Observe that by the way 14-letter blocks were marked off, one or more letters intervene between AB, BC, etc.) In contrast, if neither (1) nor (2) occurs in W, then the marked-off sequence of blocks in W can include at most four blocks of type (i) for each possible block whose square could appear in a 14-letter word, together with at most four blocks of type (ii), for a total of at most $3^7 \cdot 4+4$; then the length of W is at most $(14+1)(3^7 \cdot 4+4)+13$. Any longer word must encounter U_Δ. □

7.4. Remark. In the proof just concluded, it is possible to substitute 27 for 3^7: observe that the proof of Lemma 7.3 shows that a word of length 14 or more on $\{0, 1, 2\}$ whose transition digraph omits an edge actually must contain a square of length 8 or less. There are only 27 squares of length at most 8 that properly contain no squares. Then in the proof of Theorem 1.1, under case (i) only these 27 possibilities need be counted.

7.5. Remark. Because U_Δ is unavoidable on three letters, there is some maximum length of three-letter words that avoid U_Δ. This maximum length is at least 91, as can be seen from the fact that the word $0^{13}(10)^{13}(2010)^{13}$ avoids U_Δ.

7.6. Remark. A shorter but less computationally explicit version of the proof of Theorem 1.1 is as follows. It suffices to show that each ω-word Ω on three letters encounters U_Δ. By Lemma 7.3, a sufficiently long finite block of Ω must either (i) contain a square (of bounded length, and hence of one of a finite number of possible kinds) or (ii) must have all six two-letter pairs. In Ω, then, we can find either the same squared block AA repeated infinitely many times, or all six possible transitions infinitely many times. In the first case, we can find five separated blocks AA to produce an encounter with U_Δ; in the second case, we can find singleton blocks for A, B, C with AB, BC, etc., separated and occurring in the required order.

8. The growth rate of words avoiding U_Δ

Theorem 1.5 asserts the existence of a polynomial upper bound for the number of words of length L on four letters avoiding U_Δ, as a function of L, and also the existence of a quadratic lower bound. We first consider the lower bound (Theorem 8.2). In Corollary 5.5 it was shown that $\Omega_{2,1}$ avoids every locked word, including U_Δ, and the proof of Theorem 8.2 generalizes the construction of $\Omega_{2,1}$ to produce many long finite words of the same type. To bound the growth rate from above (Theorem 8.10) we show that every long word avoiding U_Δ has a standard form reminiscent of the generalized construction.

8.1. Lemma. *For L a power of 2 with $L \geq 8$, on a four-letter alphabet there are at least $\frac{9}{2}L^2$ words of length L that avoid every locked word.*

Proof. For the alphabet $\Gamma = \{0, 1, 2, 3\}$, define four endomorphisms H_0, H_1, H_2, H_3 of Γ^+ with values on generators as indicated in Fig. 4, where $H_i(j)$ is shown in the same position in Fig. 4 as j is shown in C_4 of Fig. 3. Define a set S_k of words of length 2^k for each k by $S_1 = \{01\}$ and $S_{k+1} = \bigcup_i H_i(S_k)$. Here are some observations for an arbitrary W in S_k:

(1) $W = \ldots 01 \ldots$, with 01 in the precise middle of W.

(2) $\mathrm{TDG}(W)$ is a subgraph of C_4. Indeed, if W' is any word with $\mathrm{TDG}(W')$ a subgraph of C_4, then $H_i(W')$ has the same property, for $i = 0$, 1, 2, 3.

$$\begin{tikzcd} 10 \arrow[r, dash] \arrow[d, dash] & 12 \arrow[d, dash] \\ 32 \arrow[r, dash] & 30 \end{tikzcd}$$
Values of H_0

$$\begin{tikzcd} 30 \arrow[r, dash] \arrow[d, dash] & 10 \arrow[d, dash] \\ 12 \arrow[r, dash] & 32 \end{tikzcd}$$
Values of H_1

$$\begin{tikzcd} 30 \arrow[r, dash] \arrow[d, dash] & 12 \arrow[d, dash] \\ 32 \arrow[r, dash] & 10 \end{tikzcd}$$
Values of H_2

$$\begin{tikzcd} 30 \arrow[r, dash] \arrow[d, dash] & 12 \arrow[d, dash] \\ 10 \arrow[r, dash] & 32 \end{tikzcd}$$
Values of H_3

Fig. 4.

(3) In W, either the letters with odd indices alternate or the letters with even indices alternate (in each case, $\ldots 0 \cdot 2 \cdot 0 \cdot 2 \ldots$ or $\ldots 1 \cdot 3 \cdot 1 \cdot 3 \ldots$), depending on which H_i was applied last.

(4) W avoids every locked word. Indeed, the proof of Theorem 5.4 with $r=0$, $m=2$, $z=1$ generalizes to the case of W. (Take residues not of letters but of their positions; the alternation mentioned in (3) serves to provide an equivalent of condition (c) of Theorem 5.4.)

(5) For $k \geq 3$, if we write $W = H_i(W')$ for $W' \in S_{k-1}$, i can be uniquely identified by the letters adjacent to the middle 01 of W as follows. If $W = \ldots 1012 \ldots$ then $i=0$; if $W = \ldots 3010 \ldots$ then $i=1$; and if $W = \ldots 3012 \ldots$ then $i=2$ if 0 and 2 alternate in W and $i=3$ if 1 and 3 alternate. (By (1) and (2), there are no other possibilities for the first letters of W, and by (4), W is square-free so that not both kinds of alternations can occur; therefore i is unique.)

(6) W can be uniquely identified merely from a knowledge of which pairs of positions in W have identical letters and which pairs of positions have distinct letters. Indeed, such knowledge tells which letters are 0, 1, and the alternation tells the positions of one of 2 and 3 and hence of the other.

Now, S_2 has three elements. By observation (5), $|S_{k+1}| = 4|S_k|$, so that by induction $|S_k| = 3 \cdot 4^{k-2}$. By observation (6), no two members of S_k differ by a permutation of the alphabet $\{0, 1, 2, 3\}$, so that there is no overlap among the twenty-four sets of words obtained from S_k by permuting the alphabet. Thus such permutations yield $24 \cdot 3 \cdot 4^{k-2} = \frac{9}{2}4^k = \frac{9}{2}L^2$ distinct words of length L that avoid every locked word, where $L = 2^k$. □

For a word U and positive integers r, L, let $N_{U,r}(L)$ be the number of words of length L on an r-letter alphabet that avoid U.

8.2. Theorem. *If U is any locked word, the function $N_{U,4}(L)$ of L has a lower bound that is quadratic in L.*

Proof. $N_{U,4}(L) > \frac{9}{8}L^2$ for all $L \geq 8$, an inequality obtained from Lemma 8.1 by (generously) reducing L to the preceding power of 2. □

8.3. Remarks. (a) By (1) of the proof of Lemma 8.1, $W = H_{i(k)}(\ldots H_{i(2)}(01)\ldots)$ can be obtained by successive extensions 01, $H_{i(k)}(01)$, $H_{i(k)}(H_{i(k-1)}(01)), \ldots, W$, in which each word is the middle portion of the next. It is not generally the case, however, that $H_i(W')$ extends W' for W in S_k, and therefore W cannot be obtained by successive extensions in the "natural" order whereby $H_{i(2)}$ is applied first.

(b) Other choices of $H_0, \ldots, H_3$ are possible. The choice made in Lemma 8.2 will be especially useful in Section 9. By another choice of $H_0, \ldots, H_3$ we could arrange instead to have each word start with 01 and be the first half of the next. Still a different choice of endomorphisms will be used below in the proof of Lemma 8.9, where the goal is again different.

We now turn to the bound on the growth rate from above. Let a *two-way* 4*-cycle* in a directed graph mean an embedded subgraph isomorphic to the graph C_4 of Fig. 3. The plan is to show that initial and final segments of any long word on four letters that avoids U_Δ have transition digraphs that *contain* two-way 4-cycles and then to show that the remaining, middle portion has a transition digraph that *is contained in* a two-way 4-cycle. We are then able to decompose long words recursively and count them.

8.4. Lemma. *For every ω-word Ω on four letters that avoids U_Δ,* TDG(Ω) *contains a two-way* 4*-cycle.*

Proof. Let us say that a word or ω-word is *compatible* with a directed graph G if its transition digraph is a subgraph of G, or equivalently, if the word can be realized as a directed path on G. We first outline an argument to show that if G is a directed graph that does not contain a two-way 4-cycle and if W is a word of length at least 18 compatible with G, then either W encounters a square or, for some nonempty words A, B, C, W has blocks AB, BC, CA, BA, and AC (although perhaps overlapping and perhaps not in the order required for an encounter with U_Δ). For this it clearly suffices to take G to be a loopless directed graph that (i) is maximal with respect to the property of not containing of two-way 4-cycle, (ii) does not contain five of the possible six directed edges on any three vertices, and (iii) is two-way connected (i.e. for any two vertices v, w, there exists a directed path from v to w). Figure 5 shows all such directed graphs, up to isomorphism. A case-by-case

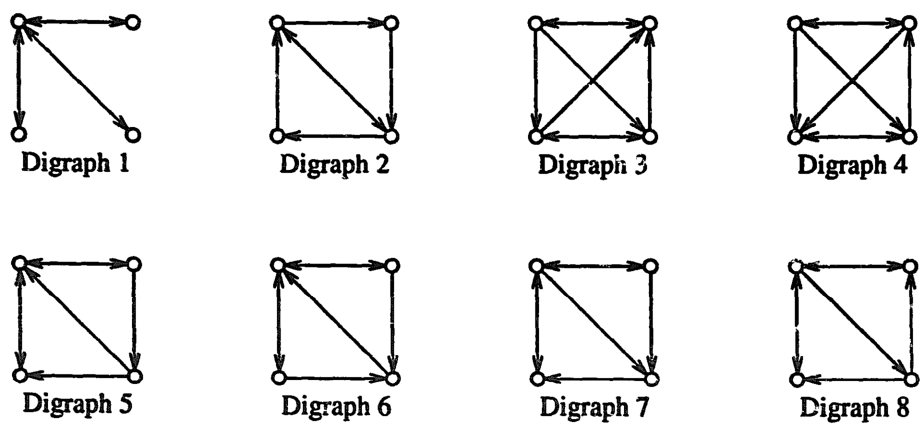


Fig. 5.

analysis (omitted here) shows that all words of length 18 that can be traced in the directed graphs of Fig. 5 contain either a square or the five fragments listed above, for some A, B, C. It follows that any infinite word compatible with G and not encountering U_Δ contains the same squared block infinitely many times or the same five fragments infinitely many times and so does encounter U_Δ (as in Remark 7.6), a contradiction. □

8.5. Corollary. *There is a bound N such that, on a four-letter alphabet, for every word W that has length N or greater and avoids U_Δ,* TDG(W) *contains a two-way 4-cycle.*

We need a simple way to refer to the three possible two-way 4-cycles that can be formed with vertices 0, 1, 2, 3. Let us use this method: in the complete directed 4-graph K_4 of Fig. 6, color solid edges one color, dashed edges another color, and dotted edges a third color. A two-way 4-cycle in K_4 can then be referred to by the two colors of the edges it uses—its "color scheme". For convenience, let us refer to a color simply by mentioning an edge of that color, e.g. "the color of 01" (or 12 or 02).

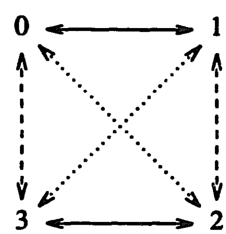


Fig. 6. K_4.

8.6. Lemma. *Let W be a (finite but long) word on letters* 0, 1, 2, 3 *that avoids U_Δ. Suppose that in W five separated N-blocks are designated. It is impossible for the transition digraphs of the first two N-blocks to have two-way 4-cycles of one color scheme and for the last three N-blocks to have two-way 4-cycles of another color scheme.*

Proof. Without loss of generality, suppose that the first two blocks have two-way 4-cycles with the colors of 01 and 12 and that the last three blocks have two-way 4-cycles with the colors of 01 and 02. Then in the five blocks in order we can find 01, 12, 20, 10, 02, and so an encounter with U_Δ. □

8.7. Lemma. *Let W be a word on letters* 0, 1, 2, 3 *that has length at least* $11N+11$ *and that avoids U_Δ. Let V be the block that remains after the first* $4(N+1)$ *letters and the last* $7(N+1)$ *letters of W have been deleted. Then* TDG(V) *is contained in a two-way 4-cycle.*

Proof. Mark off blocks of lengths N, 1, N, 1, N, 1, N, 1 at the beginning of W and blocks of lengths 1, N, 1, N, 1, ..., N (with seven occurrences of N) at the end. Of the first four initial N-blocks, some two must have two-way 4-cycles with the same color scheme, and of the last seven some three must have two-way 4-cycles with the same color scheme. By Lemma 8.6, the two at the beginning and the three at the end must have the *same* color scheme, say the colors of 01 and 12. If any transition of the remaining color occurs in V, then an encounter with U_Δ can be obtained by using the first, third, fourth, and fifth of the five blocks. For example, if 02 occurs in V, then we can find 10 in the first block, then 02, then 21 in the third block, 01 in the fourth, and 12 in the fifth, all separated; other cases are similar. Likewise, no "loop" transition such as 00 can occur in V. For example, if 00 occurs in V, then taking $A=3$ and $B=C=0$ we can find 30 in the first block, then 00, then 03 in the third block, 03 in the fourth, and 30 in the fifth, all separated; again, other cases are similar. □

Of course, in Lemma 8.7 if the length of W is at least $12N+11$ then V has length at least N and so by Corollary 8.5, TDG(V) is precisely a two-way 4-cycle.

8.8. Corollary. (i) *If Ω is an ω-word on* $\{0,1,2,3\}$ *that avoids U_Δ and Ξ is the ω-word remaining after the first* $4(N+1)$ *letters of Ω have been deleted, then* TDG(Ξ) *is a two-way* 4*-cycle.*

(ii) *If Ω is a* $\mathbb{Z}$*-word on* $\{0,1,2,3\}$ *that avoids U_Δ then* TDG(Ω) *is a two-way* 4*-cycle.*

8.9. Lemma. *If V is a word of even length with* TDG(V) *contained in a two-way* 4*-cycle, then $V=h(W')$ for some word W' and some endomorphism h of Γ^+ for* $\Gamma=\{0,1,2,3\}$ *for which each of* $h(0)$, $h(1)$, $h(2)$, $h(3)$ *is a* 2*-block. Six such endomorphisms suffice for all cases.*

Proof. Suppose that the two-way 4-cycle has edges of the colors of 01 and 12. If V has first letter 0 or 2, let $h(0)=01$, $h(1)=21$, $h(2)=03$, $h(3)=23$. (Thus h is $h_{2,1}$ of Lemma 3.2 and the example preceding Construction 3.1.) If V has first letter 1 or 3, let the values of h be these blocks reversed. The other two possible pairs of colors can be treated similarly, via permutations of the alphabet. □

8.10. Theorem. *For $U=U_\Delta$, the function $N_{U,4}(L)$ of L has an upper bound that is a polynomial function of L.*

Proof. For any sufficiently long word W avoiding U_Δ, by Lemma 8.7 we can write W as $D_0V_0E_0$, where D_0 has length at most $4(N+1)$, V_0 has even length, TDG(V_0) is contained in a two-way 4-cycle, and E_0 has length at most $7(N+1)+1$ (the extra 1 being needed to ensure that V_0 has even length). Using Lemma 8.9, write $V_0=h_1(W_1)$. Then W_1 is again a word avoiding U_Δ and can be decomposed as $W_1=D_1V_1E_1$. Continuing in this way, we obtain a decomposition

$$W=D_0h_1(D_1h_2(D_2\ldots(h_k(D_kV_kE_k))\ldots E_2)\ldots E_1)E_0,$$

in which each h_i is one of the six possibilities mentioned in Lemma 8.9 and in which V_k is empty. (There is some flexibility in splitting up W_k as D_kE_k. If V_{k-1} is shorter than four letters, h_k can be chosen arbitrarily.) This decomposition can in a broad sense be regarded as an analog of a mixed base representation with digits D_i, E_i.

From this decomposition, it is evident that the number of possible words W that involve k levels of endomorphisms is at most $4^{kM}6^k$, where $M = 11(N+1)+1$. Here the first factor represents choices of the D_i and E_i and the second represents choices of the h_i. Now, the length $|W|$ of W is at least 2^k, even if some of the D_i or E_i are empty. Therefore the logarithm of the number of choices is at most $k(\log 6 + M \log 4) \leq (\log|W|/\log 2)(\log 6 + M \log 4) \leq (\log|W|)(2M+3)$, and the number of choices is bounded by $|W|^{2M+3}$. In other words, for $U = U_\Delta$, $N_{U,4}(L) \leq L^{2M+3}$, a polynomial function. □

Remark. The counting argument of the last paragraph could be replaced by a generalization of the proof of Theorem 4.5 in [14], where a count is made of a similar set of canonical forms, but for one endomorphism instead of several.

9. The topology of a space of infinite words

It is natural to investigate the structure of the space K of $\mathbf{Z}$-words on $\{0,1,2,3\}$ that avoid U_Δ, because by Corollary 8.8(ii) the members of K have very regular transition digraphs, namely two-way 4-cycles. Some members of K can be constructed as follows. In the proof of Lemma 8.1, a set S_k of words of length 2^k was constructed for each $k \geq 1$, so that, as noted in Remarks 8.3, the words of S_k form the middle portions of the words of S_{k+1}. By iterating the construction indefinitely, we obtain a set S_∞ of $\mathbf{Z}$-words. More concretely, if S_{k+1} is mapped to S_k by taking middle 2^k-blocks, for each k, then S_∞ is the inverse limit [12, p. 91] of the sequence $\cdots \to S_3 \to S_2 \to S_1$. It is convenient to index members of each S_k so that the middle 01 occurs at indices 0, 1; then the same indexing applies to members of S_∞. Let us call indices 0, 1 the "middle" of a $\mathbf{Z}$-word. In applying an endomorphism h to a $\mathbf{Z}$-word $\Omega = \ldots w_{-1}w_0w_1\ldots$ we shall usually assume that $h(w_0)$ ends at index 0, so that in effect the middle is fixed.

9.1. Lemma. *Let Ω be a $\mathbf{Z}$-word on* $\{0,1,2,3\}$ *that avoids U_Δ. Then* (a) *Ω contains no square of length* 4, *and* (b) *in either the even-indexed positions or the odd-indexed positions of Ω, two letters alternate.*

Proof. For (a), without loss of generality suppose that the square is 0101 and that the two-way 4-cycle TDG(Ω) has edges the colors of 01 and 12 (as in C_4 of Fig. 3). Then we can group the letters of Ω into pairs 01, 21, 03, 23 and write $\Omega = h_{2,1}(\Omega')$ (although the middle may not be fixed), where Ω' then has a block 00. But Ω' again avoids U_Δ and so TDG(Ω') is a two-way 4-cycle and is loopless, a contradiction.

For (b), assume the same color scheme and use the same representation $\Omega = h_{2,1}(\Omega')$. If the two-way 4-cycle TDG(Ω') has edges the colors of 01 and 02, then 0 and 2 alternate in Ω, or if the colors of 12 and 02, then 1 and 3 alternate. The color scheme with colors of 01 and 02 cannot occur; if it did, then the edge 02 would occur as part of a block 023 or 0201, whose images under $h_{2,1}$ contain squares of length 4, or else Ω' would have a block 0202, in contradiction of (a) as applied to Ω' in place of Ω. $\square$

Remark. Actually, Ω has the stronger properties mentioned in Corollary 9.4 below.

9.2. Lemma. *Let Ω be a $\mathbb{Z}$-word such that* (a) *Ω avoids U_Δ,* (b) *Ω has transition digraph C_4 (as opposed to a different color scheme), and* (c) *Ω has* 01 *in the middle. Then $\Omega = H_i(\Omega')$ for a unique endomorphism H_i among H_0, H_1, H_2, H_3 of Fig.* 4 *and for a unique $\mathbb{Z}$-word Ω' itself satisfying* (a), (b), (c).

Proof. Since Ω satisfies (b), for any $i = 0, 1, 2, 3$ we could group the letters of Ω into pairs 10, 12, 30, 32 and write $\Omega = H_i(\Omega')$ for some Ω', and by (c) and the parities involved in the values of H_i we can even arrange to have the middle fixed. Let us do this but for a careful choice of i. If the middle 01 of Ω occurs as ...1012..., then use $i = 0$, if as ...3010..., then use $i = 1$, and if as ...3012... then use H_2 if 0 and 2 alternate, and H_3 if 1 and 3 alternate. (By Lemma 9.1, these are the only possibilities for the middle of Ω.) Then at least (a) and (c) hold for Ω'. By Corollary 8.8(ii) for Ω', TDG(Ω') is also a two-way 4-cycle. By (c) for Ω' this two-way 4-cycle at least has some edges of the color of 01. If other edges have the color of 12 then (b) holds for Ω' and we are done. By way of contradiction, suppose that other edges have the color of 02 instead. Then in the cases where H_0 or H_1 was chosen, it can be seen that Ω would have no alternations; in the cases where H_2 or H_3 was chosen, the alternations are incorrect for the choice. For the uniqueness, it suffices to observe that for each i the middle of the image under H_i of a $\mathbb{Z}$-word satisfying (b) can fit only the case we associated with H_i. $\square$

9.3. Lemma. *S_∞ is precisely the set of $\mathbb{Z}$-words that* (a) *avoid U_Δ,* (b) *have transition digraph C_4 (as opposed to a differently labeled two-way 4-cycle), and* (c) *have* 01 *at positions* 0, 1.

Proof. By (4), (2) and (1) of the proof of Lemma 8.1, the members of S_∞ fit this description. Suppose conversely that Ω is a $\mathbb{Z}$-word satisfying (a), (b), (c). The construction of Lemma 9.2 can be applied repeatedly, starting with Ω, to obtain a sequence of endomorphisms $H_{i(1)}$, $H_{i(2)}, \ldots$. Any block of Ω of length 2^k with middle at indices 0, 1 is of the form $H_{i(1)}(H_{i(2)}(\ldots H_{i(k-1)}(01)\ldots))$ and is therefore a member of S_k. In fact, Ω is the "union" of such blocks, each of which is a block of the next as k increases, and so is in S_∞. $\square$

The set K of $\mathbb{Z}$-words avoiding U_Δ is a topological space, as an inverse limit or (what is the same thing) as a closed subspace in the product topology on $\Gamma^{\mathbb{Z}}$, where

$\Gamma = \{0, 1, 2, 3\}$ as a discrete space. Recall the assertion of Theorem 1.6 that K is a Cantor space and in particular is perfect.

Proof of Theorem 1.6. K is the union of twenty-four copies of S_∞, one for each permutation of Γ. It therefore suffices to show that S_∞ is a Cantor space. Each member of S_∞ is determined by a sequence $H_{i(1)}, H_{i(2)}, \ldots$, and such sequences are arbitrary; thus S_∞ is in one-to-one correspondence with the Cantor space $\{0, 1, 2, 3\}^\omega$. The correspondence is a homeomorphism, since $\mathbb{Z}$-words agreeing in the middle 2^{k+1} letters correspond to sequences agreeing in the first k symbols, for each k. □

9.4. Corollary. *Each $\mathbb{Z}$-word Ω that avoids U_Δ is square-free and avoids every locked word.*

Proof. It suffices to examine the words in S_∞; here the assertion follows from (4) of the proof of Lemma 8.1.

10. Open questions

(1) Do there exist words that are $(n+1)$-avoidable but not n-avoidable for $n \geq 4$? Possible candidates are words W_k defined as follows. Let $U_i = a_i a_{i+1} \ldots a_n a_1 \ldots a_{i-1}$ and for each k let $W_k = U_1 x_1 U_2 x_2 \ldots x_{n-1} U_n$.

(2) For $U = U_\Delta$, is the asymptotic growth rate of $N_{U,4}(L)$ in fact polynomial, like the bounds derived in Section 8, as opposed to, say, $L^2 \log L$? If so, what is the exponent? If not, what is the infimum of s such that L^s is an upper bound?

(3) By Remark 5.7 and the representation in the proof of Lemma 8.6, U_Δ and *abcdbcabd* are not simultaneously avoidable by an ω-word on four letters. Develop a theory of simultaneous avoidability on n letters. (Zimin [35] presents part of such a theory but without a fixed number of letters, an important element. Mel'ničuk [17] gives a result for all words of length at least 2^n on an n-letter alphabet. Kobayashi [14, p. 180] mentions simultaneous avoidance of patterns as an example in an even more general framework.)

(4) Avoidance of a particular finite set of words as blocks (rather than patterns) is an important concept [11, 14]. A common generalization with (3) would be avoidance of a set of words some of whose symbols are "variables" for which blocks can be substituted and the remainder of whose symbols are "constants" that represent themselves. For example, a word U on an alphabet including the letters 0, 1 would avoid the word $0xx1$ if U does not contain 0, then a repeated block, and then 1. Develop a theory of avoidance in this sense. (Again, Kobayashi [14] presents an even more general framework.)

(5) Determine whether the space of ω-words on a four-letter alphabet that avoid U_Δ is perfect (as is the space of $\mathbb{Z}$-words studied in Section 9).

(6) Generalize the results obtained here for U_Δ to a more general class of pattern words.

(7) Adapt the method of Zimin [35] and Section 6 to replace the bound in Theorem 1.4 by a linear bound analogous to that of Theorem 1.3.

(8) Improve the bound of Theorem 1.3, perhaps by combining the congruence and marker ideas of Sections 4–6.

(9) Is there a fixed constant k such that for each finite alphabet every sufficiently long word on that alphabet is k-avoidable? Is there a list $U_0, U_1, \ldots$ of k-avoidable words such that for each finite alphabet each sufficiently long word on that alphabet encounters some U_i?

(10) Write $U \leq W$ if U encounters W or the reversal of W. This relation is a quasi-order, and factoring out the resulting equivalence relation gives a partial order. (a) Let $\mu(W)$ be as in Section 1. For avoidable W, is there an infinite antichain in the set of words on $\mu(W)$ letters such that each member of the antichain avoids W? (b) On an r-letter alphabet, let U_r be the collection of all unavoidable words. According to results in [3] and [35] (cf. [25]), U_r is finite and bounded as a partially ordered set. Up to isomorphism, which partially ordered sets arise in this way?

(11) For alphabets Σ_1 and Σ_2, the relation "W avoids U" induces a polarity between Σ_1^+ and Σ_2^+ and a resulting Galois correspondence between closed subsets [5, p. 122, 16, p. 51]. Study this construction.

Acknowledgment

The results of the paper were in part motivated by examples computed at the University of Colorado at Boulder and at the University of California, Los Angeles. The authors are especially grateful to the support staff of the Engineering Computer Network at the University of Colorado for valuable assistance.

Note added in proof. A.G. Dalalyan constructed a word that is 4-avoidable but not 3-avoidable in "Word eliminability," *Dokl. Akad. Nauk Armen. SSR* **78** (4) (1984) 156–158. A.N. Petrov has shown that scrambled words (as in Example 5.6) are 4-avoidable in the paper, "A sequence avoiding any complete word," *Mat. Zametki* **44** (4) (1988) 517–522. I. Mel'nichuk has recently announced improvements of our Theorem 1.3 and Theorem 1.4 to the bounds $\mu < \alpha + 6$ and $\mu < R + 8$ respectively.

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